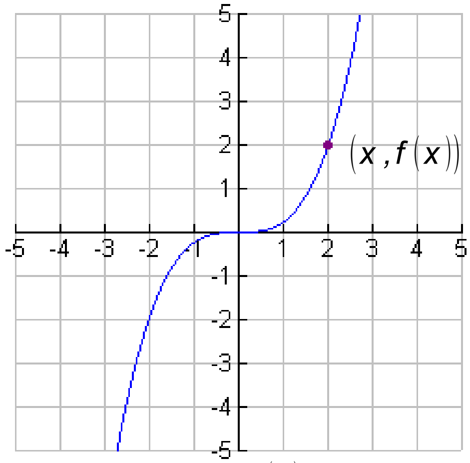
The Transformation Model for Functions

The transformation model for functions offers a powerful tool for graphing functions, obtaining equations for functions given their graph, and for adding greater insight into the nature of transformations and functions. The transformation model establishes a format for the equation of a function that makes translations, dilations, and reflections easy to identify. Once these transformations have been identified, the graph of the function can be generated by applying these transformations, *in the proper order*, to the graph of the parent function of the given function. Given the graph of a function, one can inspect the graph to establish the transformations that have been applied to it from the parent function, and an equation for the graph can be written using the transformation model.

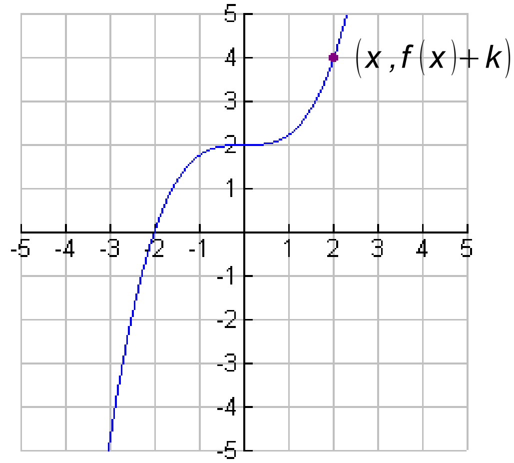
Consider a generic function $y=f(x)$. The point $(x, f(x))$ lies on the graph of the function. Now consider the following transformations that can be applied to the function $y=f(x)$, and consequently to the graph of $y=f(x)$.

Vertical Translations

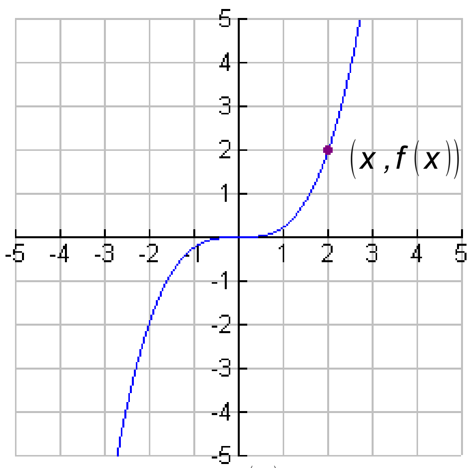
$y=f(x)+k$ causes a vertical translation of the graph of $y=f(x)$. If *k* is positive then the graph is translated *k* units upward. If *k* is negative then the graph is translated *k* units downward. Vertical translations only affect the *y*-coordinates of the points on the graph. As compared to $y=f(x)$ where the points $(x, f(x))$ were on the graph, $y=f(x)+k$ contains the points $(x, f(x)+k)$.



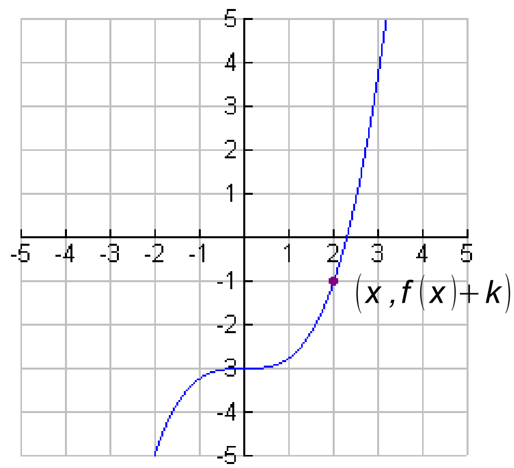
Original function $y=f(x)$



k > 0; translation k units upward



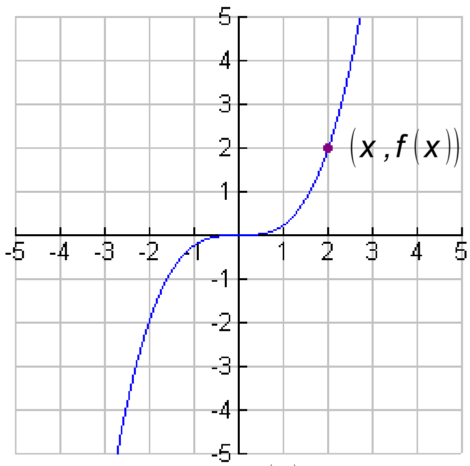
Original function $y=f(x)$



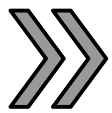
k < 0; translation k units downward

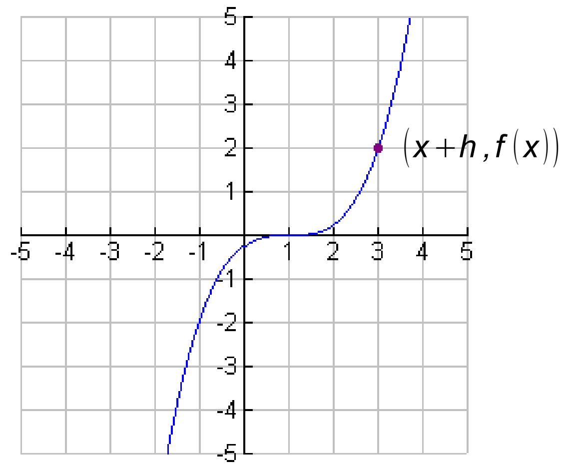
Horizontal Translations

$y=f(x-h)$ causes a horizontal translation of the graph of $y=f(x)$. If h is positive then the graph is translated h units to the right. If h is negative then the graph is translated h units to the left. Horizontal translations only affect the x-coordinates of the points on the graph. As compared to $y=f(x)$ where the points $(x,f(x))$ were on the graph, $y=f(x-h)$ contains the points $(x+h,f(x))$.

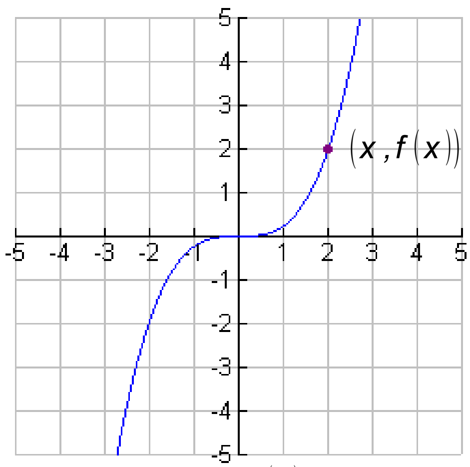


Original function $y=f(x)$

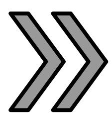


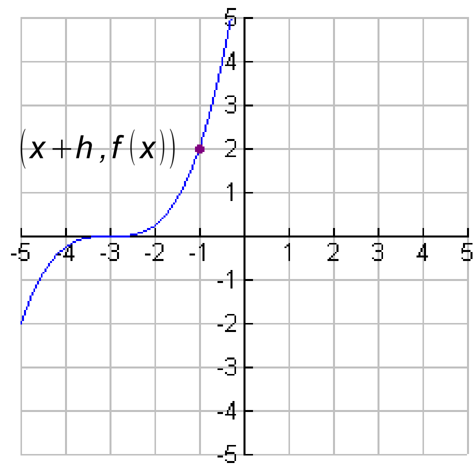


h > 0; translation h units to the right



Original function $y=f(x)$

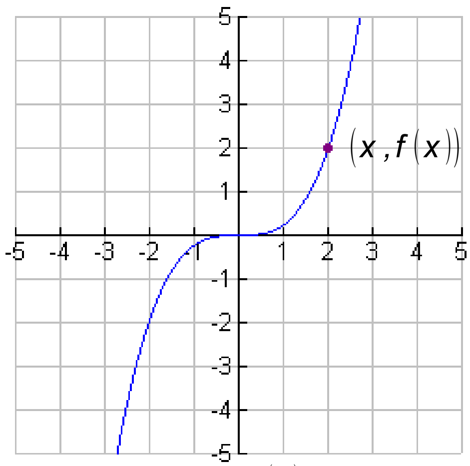




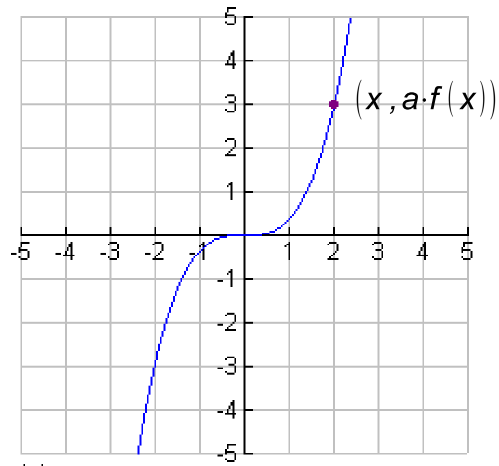
h < 0; translation h units to the left

Vertical Dilations & Reflections

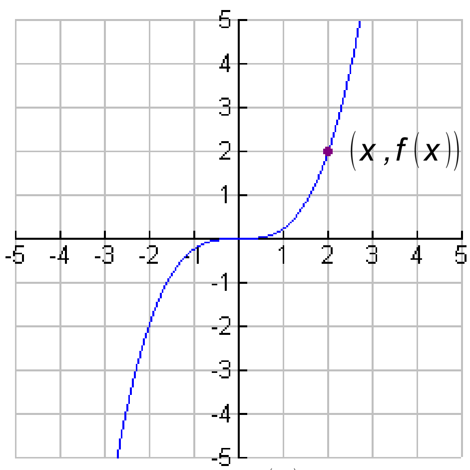
$y = a \cdot f(x)$ causes a vertical dilation of the graph of $y = f(x)$. If $|a| > 1$ the graph is stretched vertically; if $|a| < 1$ the graph is compressed vertically. In either case, the graph is vertically dilated by a scale factor of *a*. For negative values of *a* the graph is also reflected about the *x*-axis. Vertical dilations only affect the *y*-coordinates of the points on the graph. As compared to $y = f(x)$ where the points $(x, f(x))$ were on the graph, $y = a \cdot f(x)$ contains the points $(x, a \cdot f(x))$.



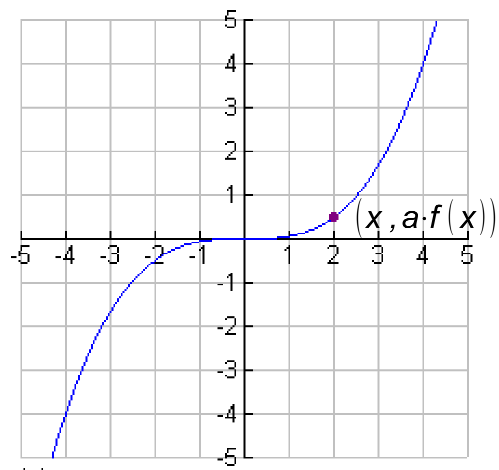
Original function $y = f(x)$



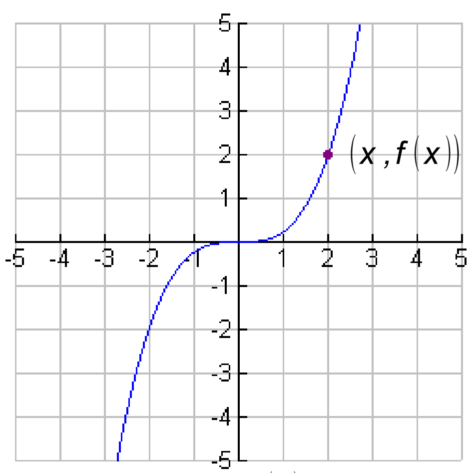
$|a| > 1$ *; vertical dilation, scale factor a*



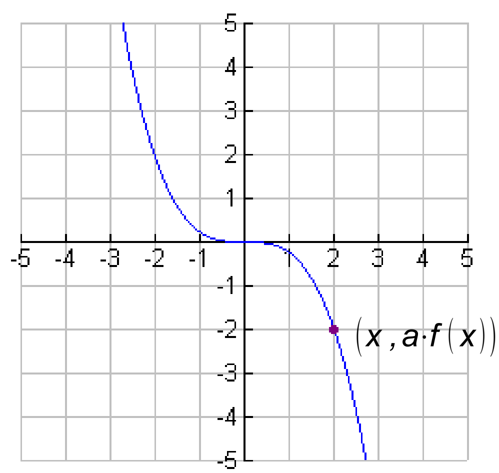
Original function $y = f(x)$



$|a| < 1$ *; vertical dilation, scale factor a*



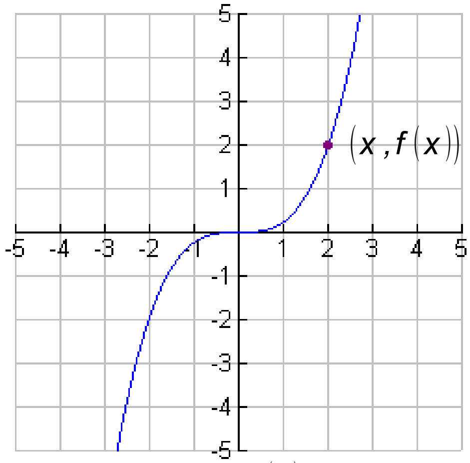
Original function $y = f(x)$



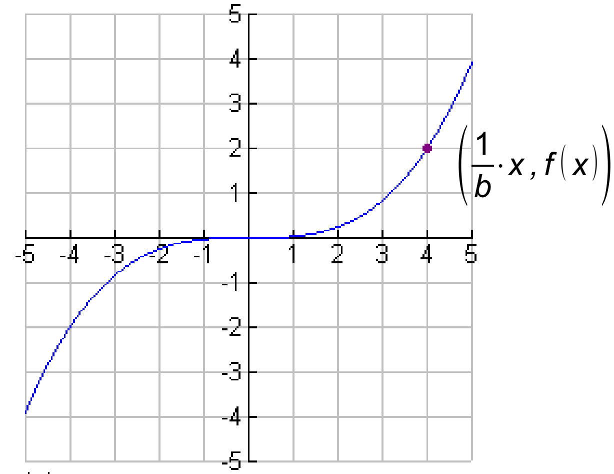
$a < 0$ *; reflection about the x-axis*

Horizontal Dilations & Reflections

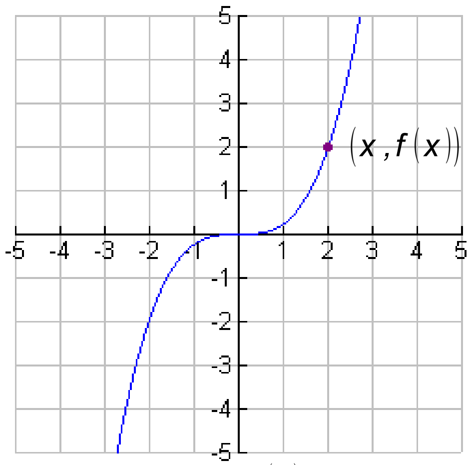
$y=f(b\cdot x)$ causes a horizontal dilation of the graph of $y=f(x)$. If $|b|>1$ the graph is compressed horizontally; if $|b|<1$ the graph is stretched horizontally. In either case, the graph is horizontally dilated by a scale factor of $1/b$. For negative values of *b* the graph is also reflected about the *y*-axis. Horizontal dilations only affect the *x*-coordinates of the points on the graph. As compared to $y=f(x)$ where the points $(x, f(x))$ were on the graph, $y=f(b\cdot x)$ contains the points $\left(\frac{1}{b}\cdot x, f(x)\right)$.



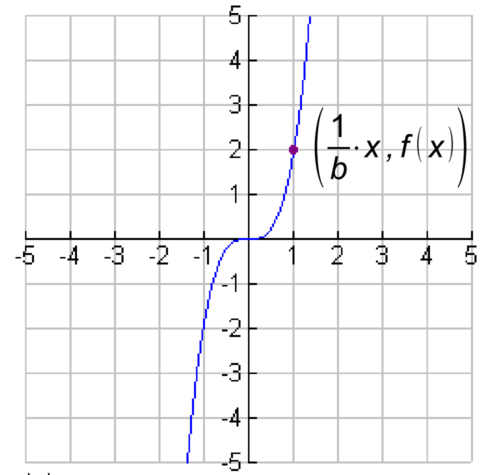
Original function $y=f(x)$



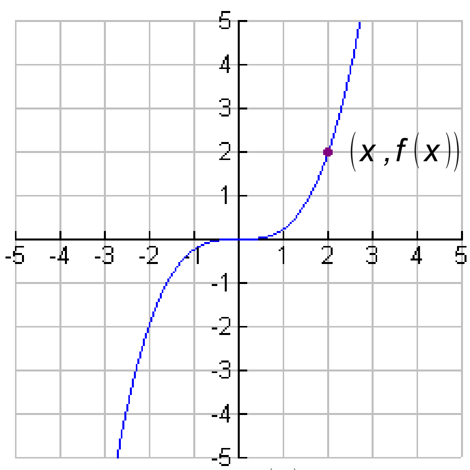
$|b|<1$ *; horizontal dilation, scale factor 1/b*



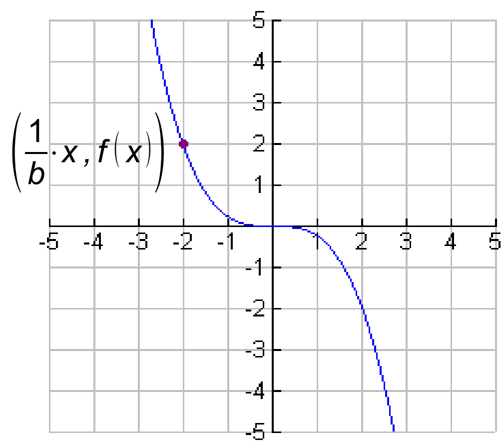
Original function $y=f(x)$



$|b|>1$ *; horizontal dilation, scale factor 1/b*



Original function $y=f(x)$



$b<0$ *; reflection about the y-axis*

Avery 2009

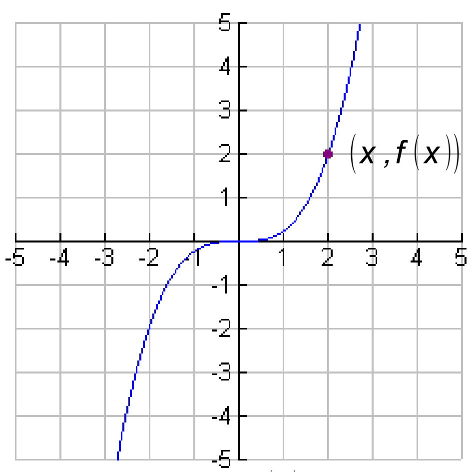
Multiple Transformations

These various transformations can be applied simultaneously within a function, giving us the following generic transformation model for a function:

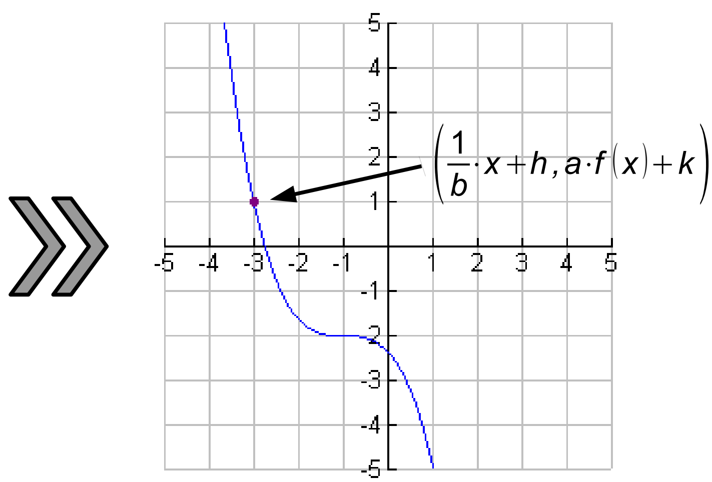
$$y = a \cdot f\left(b\left(x-h\right)\right)+k$$

Note that the parameter *b* has been factored out of the expression $\left(x-h\right)$. This is essential to ensure that the horizontal translation is properly identified. Care must also be taken with the order in which transformations are applied to the graph of the parent function in order to obtain the correct graph of the function. The order in which transformations should be applied to the parent function of a graph to generate the given function follows order or operations with regard to the action the transformation has on the coordinates of the parent function. Applying dilations and reflections to a graph results in multiplying the coordinates of graph by the appropriate scale factor for the dilation or reflection, these transformations take precedence over translations. The order in which horizontal or vertical dilations or reflections about the *x*-axis or *y*-axis are applied does not matter since they all act as multiplication operations on the coordinates of the graph. Translations should be applied to the parent function last, since translations act as addition or subtraction operations on the coordinates of the graph. The order in which horizontal or vertical translations are applied does not matter since both are addition or subtraction operations.

$y = a \cdot f\left(b\left(x-h\right)\right)+k$ would result in a vertical dilation and/or reflection, a horizontal dilation and /or reflection, a horizontal translation, and a vertical translation to the graph of $y=f\left(x\right)$. As compared to $y=f\left(x\right)$ where the points $\left(x, f\left(x\right)\right)$ were on the graph, $y = a \cdot f\left(b\left(x-h\right)\right)+k$ contains the points $\left(\frac{1}{b} \cdot x+h, a \cdot f\left(x\right)+k\right)$.



Original function $y=f\left(x\right)$



Multiple transformations

Graphing a Function Using the Transformation Model

1. Identify the parent function and key points on the graph of the parent function from the given function.
2. Identify the transformations that should be applied to the parent function based on the equation of the given function.
3. Sequentially apply the identified dilations and reflections to the key points of the parent function.
4. Sequentially apply the translations to the points that resulted from the previous step.
5. Neatly complete the graph of the given function, maintaining the proper shape and features of the graph defined by the parent function.

Obtaining the Equation for a Function From its Graph

1. Identify the parent function for the graph based on the shape or features of the graph.
2. Identify the key point of the graph that resides at the origin of the parent function (this point may not actually be on the graph of the function, as is the case with the reciprocal function).
3. Use the key point in the previous step to identify the horizontal and vertical translations of the graph from the parent function.
4. Identify another point on the graph of the function. This point will frequently be provided.
5. Use the key point in step 2 and the key point in step 4 to identify the horizontal dilation and/or reflection and the vertical dilation and/or reflection based on knowledge of the key points of the parent function.
6. Write the equation for the graph of the given function by inserting the identified transformations of the graph from the parent function in their proper location in the transformation model equation.